Eletromagnetismo I

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Problemas de Eletrostática

1 Problema das duas semi-esferas

Determine o potencial V dentro e fora de uma esfera ôca de raio unitário, com o potencial V_0 na calota superior e $-V_0$ na inferior. Supondo $V_0 = 1$, plote as equipotenciais correspondentes a $V = 0.8, 0.6, ..., -0.6, -0.8$.

Solução

```
> restart;
> with(plots): with(VectorCalculus):
> pde := expand(Laplacian(V(r, theta), 'spherical'[r, theta, phi])) = 0;
```

$$pde := \frac{2\left(\frac{\partial}{\partial r} V(r,\theta)\right)}{r} + \frac{\partial^2}{\partial r^2} V(r,\theta) + \frac{\cos(\theta)\left(\frac{\partial}{\partial \theta} V(r,\theta)\right)}{r^2 \sin(\theta)} + \frac{\frac{\partial^2}{\partial \theta^2} V(r,\theta)}{r^2} = 0 \quad \textbf{(1.1)}$$

Supondo $V(r,\theta) = R(r)\Theta(\theta)$ podemos resolver a equação exatamente

```
> V := rhs(pdsolve(pde, HINT =R(r)*Theta(theta), INTEGRATE, build));
```

$$V := \frac{\_C3 \operatorname{LegendreP}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1+4\_c_1}, \cos(\theta)\right) \_C1\, r^{\frac{1}{2}\sqrt{1+4\_c_1}}}{\sqrt{r}} + \frac{\_C3 \operatorname{LegendreP}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1+4\_c_1}, \cos(\theta)\right) \_C2\, r^{-\frac{1}{2}\sqrt{1+4\_c_1}}}{\sqrt{r}} + \frac{\_C4 \operatorname{LegendreQ}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1+4\_c_1}, \cos(\theta)\right) \_C1\, r^{\frac{1}{2}\sqrt{1+4\_c_1}}}{\sqrt{r}} + \frac{\_C4 \operatorname{LegendreQ}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1+4\_c_1}, \cos(\theta)\right) \_C2\, r^{-\frac{1}{2}\sqrt{1+4\_c_1}}}{\sqrt{r}} \quad \textbf{(1.2)}$$

> $V := expand\left(simplify\left(subs\left(\_c[1]=-\frac{1}{4}+\left(n+\frac{1}{2}\right)^2, V\right), symbolic\right)\right)$

$$V := \_C3\ \mathrm{LegendreP}(n, \cos(\theta))\ \_C1\ r^n + \frac{\_C3\ \mathrm{LegendreP}(n, \cos(\theta))\ \_C2}{r\, r^n} + \_C4\ \mathrm{LegendreQ}(n, \cos(\theta))\ \_C1\ r^n + \frac{\_C4\ \mathrm{LegendreQ}(n, \cos(\theta))\ \_C2}{r\, r^n} \quad \textbf{(1.3)}$$

Note que a função de Legendre de segunda espécie LegendreQ diverge nos pontos terminais de θ:

> $evalf(\mathrm{LegendreQ}(7, \cos(\pi)))$

```
Error, (in LegendreQ) numeric exception: division by zero
```

Portanto, devemos fazer _C4=0.

```
> V:=subs({_C4=0,_C3=1,_C1=A,_C2=B},V);
```

$$V := \mathrm{LegendreP}(n, \cos(\theta))\ A\ r^n + \frac{\mathrm{LegendreP}(n, \cos(\theta))\ B}{r\, r^n} \quad \textbf{(1.4)}$$

Soluções interior e exterior:

```
> Vin:=subs(B=0,V);Vout:=subs(A=0,V);
```

$$Vin := \mathrm{LegendreP}(n, \cos(\theta))\ A\ r^n$$

$$Vout := \frac{\mathrm{LegendreP}(n, \cos(\theta))\ B}{r\, r^n} \quad \textbf{(1.5)}$$

Os coeficientes da série que representa a solução são dados por

$$A_n = \frac{(2\,n+1)}{2}\int_{-1}^{1} f\,P_n(u)\,\mathrm{d}u$$

```
> AA:=n->((2*n+1)/2)*(int(V0*LegendreP(n,u),u=1..0)+int((-V0*
  LegendreP(n,u),u=0..-1))):
```

```
> VIN:=sum(eval(Vin,A=AA(n)),n=0..15);
```

$$VIN := -\frac{3}{2}\cos(\theta)\ V0\ r + \frac{7}{8}\mathrm{LegendreP}(3, \cos(\theta))\ V0\ r^3 - \frac{11}{16}\mathrm{LegendreP}(5, \cos(\theta))\ V0\ r^5 + \frac{75}{128}\mathrm{LegendreP}(7, \cos(\theta))\ V0\ r^7 - \frac{133}{256}\mathrm{LegendreP}(9, \cos(\theta))\ V0\ r^9 + \frac{483}{1024}\mathrm{LegendreP}(11, \cos(\theta))\ V0\ r^{11} - \frac{891}{2048}\mathrm{LegendreP}(13, \cos(\theta))\ V0\ r^{13} + \frac{13299}{32768}\mathrm{LegendreP}(15, \cos(\theta))\ V0\ r^{15} \quad \textbf{(1.6)}$$

```
> VOUT := sum(eval(Vout, B = AA(n)), n = 0 .. 15);
```

$$VOUT := -\frac{3}{2}\frac{\cos(\theta)\ V0}{r^2} + \frac{7}{8}\frac{\mathrm{LegendreP}(3, \cos(\theta))\ V0}{r^4} - \frac{11}{16}\frac{\mathrm{LegendreP}(5, \cos(\theta))\ V0}{r^6} + \frac{75}{128}\frac{\mathrm{LegendreP}(7, \cos(\theta))\ V0}{r^8} \quad \textbf{(1.7)}$$

$$-\frac{133}{256}\frac{\text{LegendreP}(9,\cos(\theta))\ V0}{r^{10}}+\frac{483}{1024}\frac{\text{LegendreP}(11,\cos(\theta))\ V0}{r^{12}}$$

$$-\frac{891}{2048}\frac{\text{LegendreP}(13,\cos(\theta))\ V0}{r^{14}}+\frac{13299}{32768}\frac{\text{LegendreP}(15,\cos(\theta))\ V0}{r^{16}}$$

Podemos plotar as equipotenciais no plano xz, fazendo $r=\sqrt{x^2+z^2}$ e $\cos(\theta)=\frac{z}{r}$, com $V_0=1$.

```
> r:=sqrt(x^2+z^2):cos(theta):=z/r:V0:=1.0:
```

O potencial pode ser especificado por

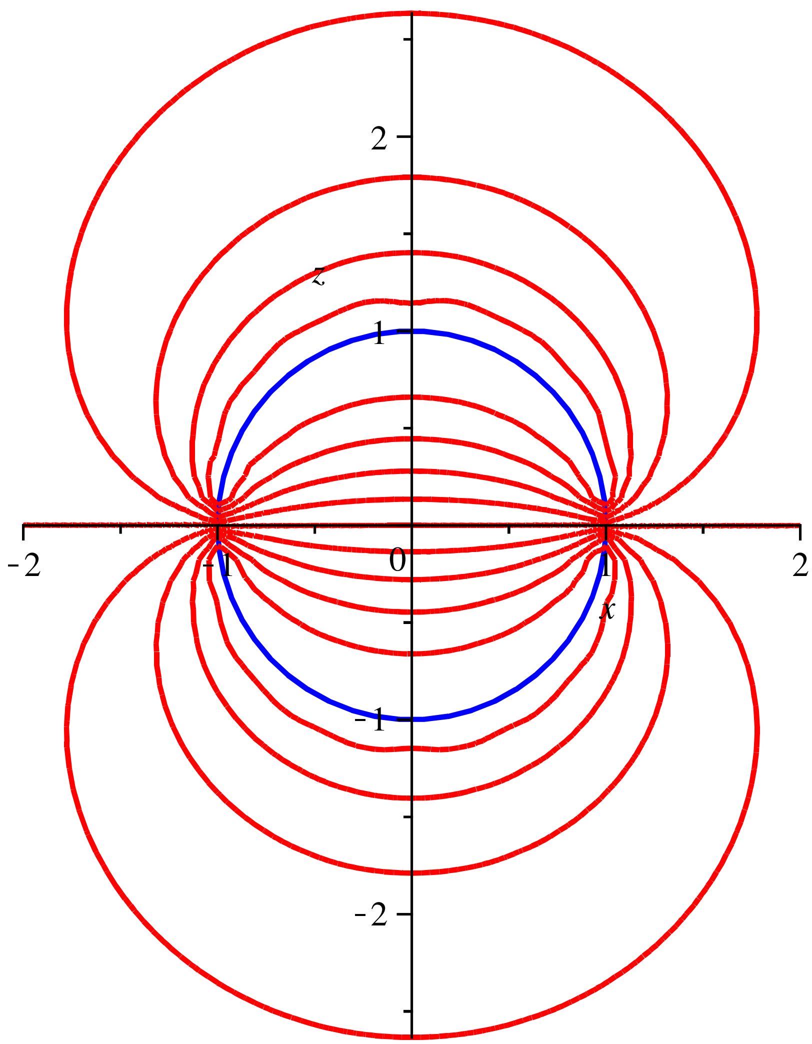
```
> Vpw:=piecewise(r<1,VIN,r>1,VOUT):
```

A esfera no plano xz é representada por

```
> with(plottools):
> c:=circle([0,0],1,color=blue,thickness=2):
```

As equipotencias são dadas por

```
> cp:=contourplot(Vpw,x=-2..2,z=-5..4,contours=[seq(0.8-0.2*i,
  i=0..8)],grid=[60,60],thickness=2,color=red,numpoints=7000):
> display({c,cp},scaling=constrained);
```



2 Variação do problema das duas semi-esferas

Determine o potencial V dentro e fora de uma esfera ôca de raio unitário, com o potencial na superfície especificado do seguinte modo: Se θ é o ângulo zenital, a superfície entre θ = 0 e π/4 tem potencial constante V_0 , a parte intermediária entre π/4 e 3π/4 tem um potencial dado por $\sqrt{2}\cos\theta\, V_0$, e a porção inferior entre 3π/4 e π com um potencial $-V_0$. Supondo $V_0 = 1$, plote as equipotenciais correspondentes a $V = 0.8, 0.6, ..., -0.6, -0.8$.

Solução

```
> restart;
> with(plots): with(VectorCalculus):
> pde := expand(Laplacian(V(r, theta), 'spherical'[r, theta, phi])) = 0;
```

$$pde := \frac{2\left(\frac{\partial}{\partial r}V(r,\theta)\right)}{r} + \frac{\partial^2}{\partial r^2}V(r,\theta) + \frac{\cos(\theta)\left(\frac{\partial}{\partial\theta}V(r,\theta)\right)}{r^2\sin(\theta)} + \frac{\frac{\partial^2}{\partial\theta^2}V(r,\theta)}{r^2} = 0 \quad \textbf{(2.1)}$$

Supondo $V(r,\theta) = R(r)\Theta(\theta)$ podemos resolver a equação exatamente

```
> V := rhs(pdsolve(pde, HINT =R(r)*Theta(theta), INTEGRATE, build));
```

$$V := \frac{\_C3\,\mathrm{LegendreP}\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\_c_1},\cos(\theta)\right)\_C1\,r^{\frac{1}{2}\sqrt{1+4\_c_1}}}{\sqrt{r}} + \frac{\_C3\,\mathrm{LegendreP}\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\_c_1},\cos(\theta)\right)\_C2\,r^{-\frac{1}{2}\sqrt{1+4\_c_1}}}{\sqrt{r}} + \frac{\_C4\,\mathrm{LegendreQ}\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\_c_1},\cos(\theta)\right)\_C1\,r^{\frac{1}{2}\sqrt{1+4\_c_1}}}{\sqrt{r}} + \frac{\_C4\,\mathrm{LegendreQ}\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\_c_1},\cos(\theta)\right)\_C2\,r^{-\frac{1}{2}\sqrt{1+4\_c_1}}}{\sqrt{r}} \quad \textbf{(2.2)}$$

$$> V := expand\left(simplify\left(subs\left(\_c[1] = -\frac{1}{4}+\left(n+\frac{1}{2}\right)^2, V\right), symbolic\right)\right)$$

$$V := \_C3\,\mathrm{LegendreP}(n,\cos(\theta))\,\_C1\,r^n + \frac{\_C3\,\mathrm{LegendreP}(n,\cos(\theta))\,\_C2}{r\,r^n} + \_C4\,\mathrm{LegendreQ}(n,\cos(\theta))\,\_C1\,r^n + \frac{\_C4\,\mathrm{LegendreQ}(n,\cos(\theta))\,\_C2}{r\,r^n} \quad \textbf{(2.3)}$$

> $evalf(\text{LegendreP}(7, \cos(\pi)))$

$$-1. \tag{2.4}$$

Note que a função de Legendre de segunda espécie LegendreQ diverge nos pontos terminais de θ:

> $evalf(\text{LegendreQ}(7, \cos(\pi)))$

```
Error, (in LegendreQ) numeric exception: division by zero
```

Portanto, devemos fazer _C4=0.

```
> V:=subs({_C4=0,_C3=1,_C1=A,_C2=B},V);
```

$$V := \text{LegendreP}(n, \cos(\theta))\, A\, r^n + \frac{\text{LegendreP}(n, \cos(\theta))\, B}{r\, r^n} \tag{2.5}$$

Soluções interior e exterior:

```
> Vin:=subs(B=0,V);Vout:=subs(A=0,V);
```

$$Vin := \text{LegendreP}(n, \cos(\theta))\, A\, r^n$$

$$Vout := \frac{\text{LegendreP}(n, \cos(\theta))\, B}{r\, r^n} \tag{2.6}$$

>

Especificamos o potencial:

```
> f1 := -V0; f2 := sqrt(2)*u*V0; f3 := V0;
```

$$f1 := -V0$$

$$f2 := \sqrt{2}\, u\, V0$$

$$f3 := V0 \tag{2.7}$$

Os coeficientes da série que representa a solução são dados por

$$A_n = \frac{(2\,n+1)}{2} \int_{-1}^{1} f\, P_n(u)\, du$$

```
> AA:=n->((2*n+1)/2)*(int(f1*LegendreP(n,u),u=-1..1/sqrt(2))+
  int(f2*LegendreP(n,u),u=-1/sqrt(2)..1/sqrt(2))+int(f3*
  LegendreP(n,u),u=1/sqrt(2)..1));
```

$$AA := n \to (2\,n+1)\, \frac{1}{2} \left(VectorCalculus{:}\text{-}int\left(f1\, \text{LegendreP}(n, u),\, u = VectorCalculus{:}\text{-} `\text{-}`(1)\, ..1\, \frac{1}{\sqrt{2}} \right) + VectorCalculus{:}\text{-}int\left(f2\, \text{LegendreP}(n, u),\, u = VectorCalculus{:}\text{-} `\text{-}`\left(1\, \frac{1}{\sqrt{2}}\right) ..1\, \frac{1}{\sqrt{2}} \right) + VectorCalculus{:}\text{-}int\left(f3\, \text{LegendreP}(n, u),\, u = 1\, \frac{1}{\sqrt{2}}\, ..1 \right) \right) \tag{2.8}$$

> $AA(1)$

$$\frac{5}{4}\, V0 \tag{2.9}$$

```
> VIN:=sum(eval(Vin,A=AA(n)),n=0..12);
```

$$VIN := -\frac{1}{2}\, V0 \left(\frac{1}{2}\sqrt{2} + 1 \right) + \frac{1}{2}\, V0 \left(1 - \frac{1}{2}\sqrt{2} \right) + \frac{5}{4} \cos(\theta)\, V0\, r \tag{2.10}$$

$$+\frac{5}{8}\,\mathrm{LegendreP}(2,\cos(\theta))\,\sqrt{2}\,V0\,r^2-\frac{7}{32}\,\mathrm{LegendreP}(3,\cos(\theta))\,V0\,r^3$$
$$+\frac{9}{64}\,\mathrm{LegendreP}(4,\cos(\theta))\,\sqrt{2}\,V0\,r^4-\frac{11}{128}\,\mathrm{LegendreP}(5,\cos(\theta))\,V0\,r^5$$
$$-\frac{91}{256}\,\mathrm{LegendreP}(6,\cos(\theta))\,\sqrt{2}\,V0\,r^6+\frac{85}{2048}\,\mathrm{LegendreP}(7,\cos(\theta))\,V0\,r^7$$
$$-\frac{459}{4096}\,\mathrm{LegendreP}(8,\cos(\theta))\,\sqrt{2}\,V0\,r^8+\frac{323}{8192}\,\mathrm{LegendreP}(9,\cos(\theta))\,V0\,r^9$$
$$+\frac{4515}{16384}\,\mathrm{LegendreP}(10,\cos(\theta))\,\sqrt{2}\,V0\,r^{10}-\frac{1219}{65536}\,\mathrm{LegendreP}(11,$$
$$\cos(\theta))\,V0\,r^{11}+\frac{12525}{131072}\,\mathrm{LegendreP}(12,\cos(\theta))\,\sqrt{2}\,V0\,r^{12}$$

```
> VOUT := sum(eval(Vout, B = AA(n)), n = 0 .. 12);
```

$$VOUT := -\frac{0.7071067810}{\sqrt{x^2+z^2}}+\frac{1.250000000\,z}{(x^2+z^2)^{3/2}} \quad \textbf{(2.11)}$$
$$+\frac{0.8838834762\,\mathrm{LegendreP}\left(2.,\frac{z}{\sqrt{x^2+z^2}}\right)}{(x^2+z^2)^{3/2}}$$
$$-\frac{0.2187500000\,\mathrm{LegendreP}\left(3.,\frac{z}{\sqrt{x^2+z^2}}\right)}{(x^2+z^2)^{2}}$$
$$+\frac{0.1988737822\,\mathrm{LegendreP}\left(4.,\frac{z}{\sqrt{x^2+z^2}}\right)}{(x^2+z^2)^{5/2}}$$
$$-\frac{0.08593750000\,\mathrm{LegendreP}\left(5.,\frac{z}{\sqrt{x^2+z^2}}\right)}{(x^2+z^2)^{3}}$$
$$-\frac{0.5027087271\,\mathrm{LegendreP}\left(6.,\frac{z}{\sqrt{x^2+z^2}}\right)}{(x^2+z^2)^{7/2}}$$
$$+\frac{0.04150390625\,\mathrm{LegendreP}\left(7.,\frac{z}{\sqrt{x^2+z^2}}\right)}{(x^2+z^2)^{4}}$$

$$- \frac{0.1584775452 \operatorname{LegendreP}\left(8., \frac{z}{\sqrt{x^2+z^2}}\right)}{\left(x^2+z^2\right)^{9/2}}$$

$$+ \frac{0.03942871094 \operatorname{LegendreP}\left(9., \frac{z}{\sqrt{x^2+z^2}}\right)}{\left(x^2+z^2\right)^{5}}$$

$$+ \frac{0.3897201070 \operatorname{LegendreP}\left(10., \frac{z}{\sqrt{x^2+z^2}}\right)}{\left(x^2+z^2\right)^{11/2}}$$

$$- \frac{0.01860046387 \operatorname{LegendreP}\left(11., \frac{z}{\sqrt{x^2+z^2}}\right)}{\left(x^2+z^2\right)^{6}}$$

$$+ \frac{0.1351396550 \operatorname{LegendreP}\left(12., \frac{z}{\sqrt{x^2+z^2}}\right)}{\left(x^2+z^2\right)^{13/2}}$$

Podemos plotar as equipotenciais no plano xz, fazendo $r = \sqrt{x^2+z^2}$ e $\cos(\theta) = \frac{z}{r}$, com $V_0 = 1$.

```
> r:=sqrt(x^2+z^2):cos(theta):=z/r:V0:=1.0:
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> display({c,cp},scaling=constrained);
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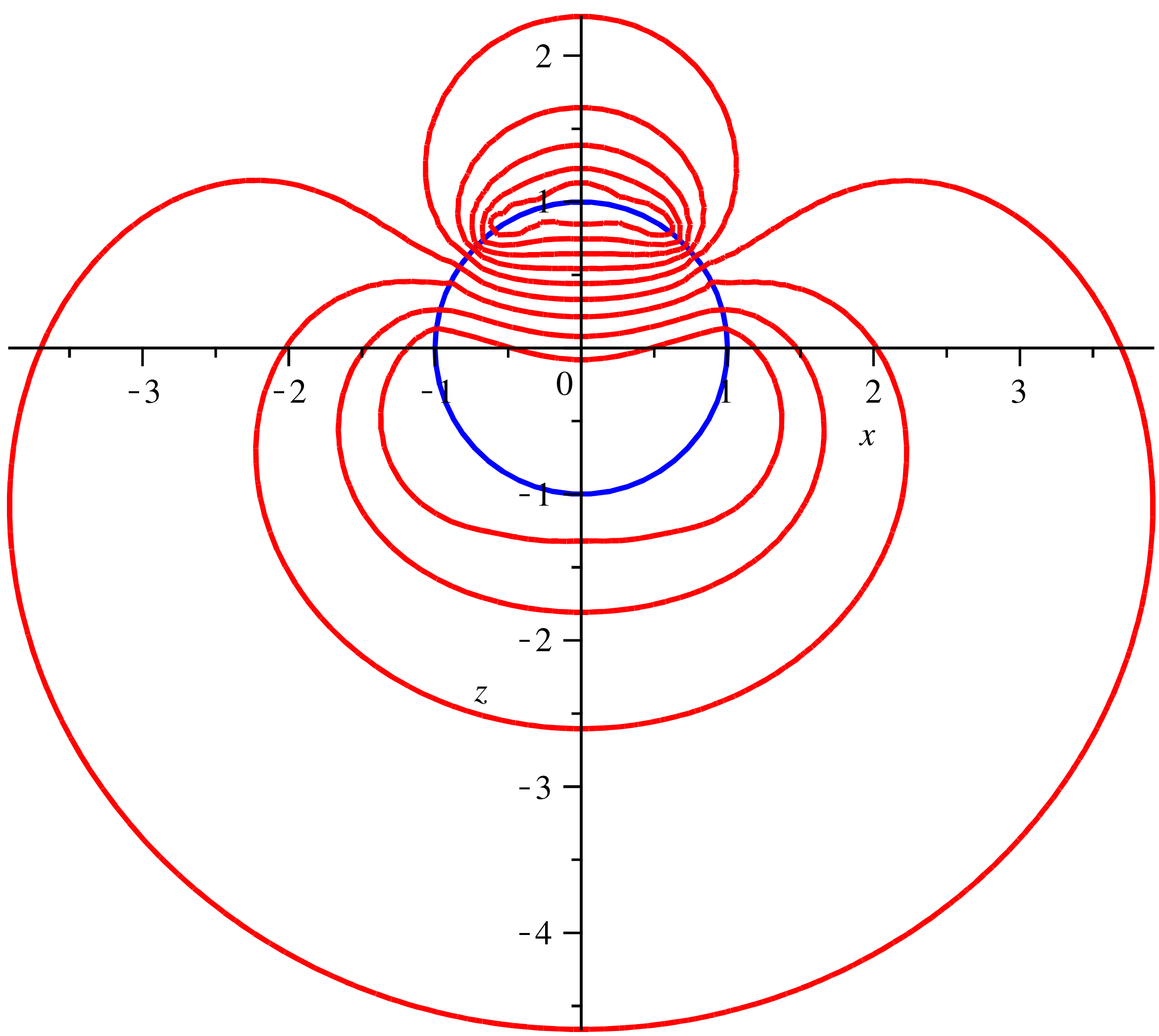

2
1
-3
-2
-1
0
1
2
3
x
-1
-2
z
-3
-4

>