

Remarks

(i) In conclusion, we point out a result that will be of special interest to us in Sections 6.5 and 6.6. Replacing the complex variable s with the usual symbol z , we see that when $k = 1$, formula (8) for the Laurent series coefficients yields $a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz$, or more important,

$$\oint_C f(z) dz = 2\pi i a_{-1}. \quad (20)$$

(ii) Regardless how a Laurent expansion of a function f is obtained in a specified annular domain it is *the* Laurent series; that is, the series we obtain is unique.

EXERCISES 6.3 Answers to selected odd-numbered problems begin on page ANS-19.

In Problems 1–6, expand the given function in a Laurent series valid for the given annular domain.

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| 1. $f(z) = \frac{\cos z}{z}, 0 < z $ | 2. $f(z) = \frac{z - \sin z}{z^5}, 0 < z $ |
| 3. $f(z) = e^{-1/z^2}, 0 < z $ | 4. $f(z) = \frac{1 - e^z}{z^2}, 0 < z $ |
| 5. $f(z) = \frac{e^z}{z - 1}, 0 < z - 1 $ | 6. $f(z) = z \cos \frac{1}{z}, 0 < z $ |

In Problems 7–12, expand $f(z) = \frac{1}{z(z-3)}$ in a Laurent series valid for the indicated annular domain.

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| 7. $0 < z < 3$ | 8. $ z > 3$ |
| 9. $0 < z - 3 < 3$ | 10. $ z - 3 > 3$ |
| 11. $1 < z - 4 < 4$ | 12. $1 < z + 1 < 4$ |

In Problems 13–16, expand $f(z) = \frac{1}{(z-1)(z-2)}$ in a Laurent series valid for the given annular domain.

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| 13. $1 < z < 2$ | 14. $ z > 2$ |
| 15. $0 < z - 1 < 1$ | 16. $0 < z - 2 < 1$ |

In Problems 17–20, expand $f(z) = \frac{z}{(z+1)(z-2)}$ in a Laurent series valid for the given annular domain.

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| 17. $0 < z + 1 < 3$ | 18. $ z + 1 > 2$ |
| 19. $1 < z < 2$ | 20. $0 < z - 2 < 3$ |

In Problems 21 and 22, expand $f(z) = \frac{1}{z(1-z)^2}$ in a Laurent series valid for the given annular domain.

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| 21. $0 < z < 1$ | 22. $ z > 1$ |
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In Problems 23 and 24, expand $f(z) = \frac{1}{(z-2)(z-1)^3}$ in a Laurent series valid for the given annular domain.

23. $0 < |z-2| < 1$

24. $0 < |z-1| < 1$

In Problems 25 and 26, expand $f(z) = \frac{7z-3}{z(z-1)}$ in a Laurent series valid for the given annular domain.

25. $0 < |z| < 1$

26. $0 < |z-1| < 1$ [Hint: $\frac{7z-3}{z} = \frac{7(z-1)+4}{1+(z-1)}$.]

In Problems 27 and 28, expand $f(z) = \frac{z^2-2z+2}{z-2}$ in a Laurent series valid for the given annular domain.

27. $1 < |z-1|$

28. $0 < |z-2|$

In Problems 29 and 30, use $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$, $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$, and long division to find the first three nonzero terms of a Laurent series of the given function f valid for $0 < |z| < \pi$.

29. $f(z) = \csc z$

30. $f(z) = \cot z$

Focus on Concepts

- 31.** The function $f(z) = \frac{1}{(z+2)(z-4i)}$ possesses a Laurent series $f(z) = \sum_{k=-\infty}^{\infty} a_k(z+2)^k$ valid in the annulus $r < |z+2| < R$. What are r and R ?
- 32.** Consider the function $f(z) = \frac{e^{-2z}}{(z+1)^2}$. Use (7) to find the principal part of the Laurent series expansion of f about $z_0 = -1$ that is valid on the annulus $0 < |z+1| < \infty$.
- 33.** Consider the function $f(z) = \frac{1}{(z-5)^3}$. What is the Laurent series expansion of f about $z_0 = 5$ that is valid on the annulus $0 < |z-5| < \infty$?

6.4 Zeros and Poles

Suppose $z = z_0$ is an isolated singularity of a complex function f , and that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k(z-z_0)^k \quad (1)$$

is the Laurent series representation of f valid for the punctured open disk $0 < |z-z_0| < R$. We saw in the preceding section that a Laurent series (1) consists of two parts. That part of the series in (1) with negative powers of $z-z_0$, namely,

$$\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} \quad (2)$$