EXERCISES 6.6

Answers to selected odd-numbered problems begin on page ANS-20.

6.6.1 Evaluation of Real Trigonometric Integrals

In Problems 1–12, evaluate the given trigonometric integral.

1. $\displaystyle\int_0^{2\pi} \frac{1}{1+0.5\sin\theta}\,d\theta$

2. $\displaystyle\int_0^{2\pi} \frac{1}{10-6\cos\theta}\,d\theta$

3. $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta}\,d\theta$

4. $\displaystyle\int_0^{2\pi} \frac{1}{1+3\cos^2\theta}\,d\theta$

5. $\displaystyle\int_0^{\pi} \frac{1}{2-\cos\theta}\,d\theta$ [*Hint*: Let $t = 2\pi - \theta$.]

6. $\displaystyle\int_0^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta$

7. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta$

8. $\displaystyle\int_0^{2\pi} \frac{\cos^2\theta}{3-\sin\theta}\,d\theta$

9. $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}\,d\theta$

10. $\displaystyle\int_0^{2\pi} \frac{1}{\cos\theta+2\sin\theta+3}\,d\theta$

11. $\displaystyle\int_0^{2\pi} \frac{\cos^2\theta}{2+\sin\theta}\,d\theta$

12. $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$

In Problems 13 and 14, establish the given general result. Use Problem 13 to verify the answer in Example 1. Use Problem 14 to verify the answer to Problem 7.

13. $\displaystyle\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2}\,d\theta = \frac{a\pi}{(\sqrt{a^2-1})^3},\ a > 1$

14. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right),\ a > b > 0$

6.6.2 Evaluation of Real Improper Integrals

In Problems 15–26, evaluate the Cauchy principal value of the given improper integral.

15. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2-2x+2}\,dx$

16. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2-6x+25}\,dx$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,dx$

18. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$

19. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\,dx$

20. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+4)^3}\,dx$

21. $\displaystyle\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}\,dx$

22. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2(x^2+9)}\,dx$

23. $\displaystyle\int_{0}^{\infty} \frac{x^2+1}{x^4+1}\,dx$

24. $\displaystyle\int_{0}^{\infty} \frac{1}{x^6+1}\,dx$

25. $\displaystyle\int_{0}^{\infty} \frac{x^2}{x^6+1}\,dx$

26. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+2x+2)(x^2+1)^2}\,dx$

In Problems 27–38, evaluate the Cauchy principal value of the given improper integral.

27. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$

28. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos\ 2x}{x^2+1}\,dx$

29. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\,dx$

30. $\displaystyle\int_{0}^{\infty} \frac{\cos x}{(x^2+4)^2}\,dx$

31. $\displaystyle\int_0^{\infty} \frac{\cos\ 3x}{(x^2+1)^2}dx$

32. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}dx$

33. $\displaystyle\int_0^{\infty} \frac{\cos\ 2x}{x^4+1}dx$

34. $\displaystyle\int_0^{\infty} \frac{x\sin x}{x^4+1}dx$

35. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+9)}dx$

36. $\displaystyle\int_0^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}dx$

37. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x+i}\,dx$ [*Hint*: First substitute $\sin x = (e^{ix} - e^{-ix})/2i$, x real.]

38. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x + x\sin x}{x^2+1}dx$ [*Hint*: Consider $e^{iz}/(z-i)$.]

In Problems 39–42, use an indented contour and residues to establish the Cauchy principal value of the given improper integral.

39. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}dx = \pi$

40. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}dx = \pi(1-e^{-1})$

41. $\displaystyle\int_0^{\infty} \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$

42. $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos x}{x^2-3x+2}\,dx = \pi[\sin 1 - 2\sin 2]$

6.6.3 Integration along a Branch Cut

In Problems 43–46**,** proceed as in Example 6 to establish the Cauchy principal value for the given improper integral.

43. $\displaystyle\int_0^{\infty} \frac{1}{\sqrt{x}(x^2+1)}\,dx = \frac{\pi}{\sqrt{2}}$

44. $\displaystyle\int_0^{\infty} \frac{1}{\sqrt{x}(x+1)(x+4)}\,dx = \frac{\pi}{3}$

45. $\displaystyle\int_0^{\infty} \frac{\sqrt{x}}{(x^2+1)^2}\,dx = \frac{\pi}{4\sqrt{2}}$

46. $\displaystyle\int_0^{\infty} \frac{x^{1/3}}{(x+1)^2}\,dx = \frac{2\pi}{3\sqrt{3}}$

In Problems 47 and 48, establish the Cauchy principal value for the given improper integral. Use Problem 47 to verify the answer in Example 6. Use Problem 48 to verify the answer to Problem 45.

47. $\displaystyle\int_0^{\infty} \frac{x^{\alpha-1}}{x+1}\,dx = \frac{\pi}{\sin\alpha\pi},\ 0<\alpha<1,$

48. $\displaystyle\int_0^{\infty} \frac{x^{\alpha}}{(x^2+1)^2}\,dx = \frac{\pi(1-\alpha)}{4\cos(\alpha\pi/2)},\ -1<\alpha<3,\ \alpha\neq 1$

Miscellaneous Real Integrals

49. Use the contour C shown in Figure 6.18 to show that

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^{x}}dx = \frac{\pi}{\sin\alpha\pi},\quad 0<\alpha<1.$$

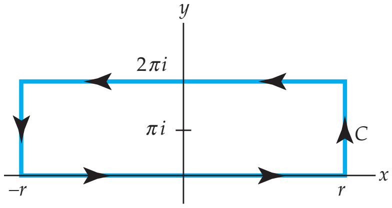

Figure 6.18 Figure for Problem 49

50. The integral result $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ can be established using elelmentary calculus and polar coordinates. Use this result, the contour integral $\oint_C e^{-z^2} e^{i\alpha z}\,dz$, and the contour C shown in Figure 6.19, to show that

$$\text{P.V.} \int_0^{\infty} e^{-x^2} \cos \alpha x \, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}.$$

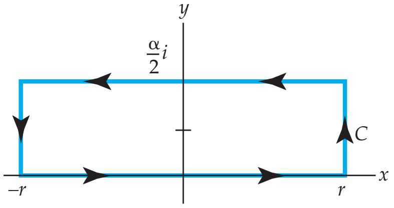

Figure 6.19 Figure for Problem 50

51. Discuss how to evaluate the Cauchy principal value of

$$\int_0^{\infty} \frac{x^{\alpha-1}}{x-1}\,dx, \ \ 0 < \alpha < 1.$$

Carry out your ideas.

52. **(a)** Use a graphics calculator or computer graphing program to plot on the same coordinates axes the graphs of $\sin\theta$ and $2\theta/\pi$ on the interval $0 \le \theta \le \pi/2$. Explain in graphical terms the validity of the inequality $\sin\theta \ge 2\theta/\pi$ on the interval $0 \le \theta \le \pi/2$. Use this inequality to prove that for $R > 0$,

$$\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta < \frac{\pi}{2R}.$$

(b) Explain how the result in part (a) leads us to conclude that for $R > 0$,

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}. \tag{48}$$

The result in (48) is known as **Jordan's inequality,** which is often useful when evaluating integrals of the form $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$ and $\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$.

53. Reconsider the integral in Problem 39 along with the indented contour in Figure 6.14. Use Jordan's inequality in Problem 52 to show that $\int_{C_R} \to 0$ as $R \to \infty$.

54. Investigate the integral $\int_0^{2\pi} \frac{1}{a - \sin\theta}\,d\theta$, $|a| \le 1$, in light of the evaluation procedure outlined in Subsection 6.6.1.

55. Use Euler's formula as a starting point in the evaluation of the integral

$$\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta - n\theta) + i\sin(\sin\theta - n\theta)]\,d\theta, n = 0, 1, 2, \ldots .$$

56. From your work in Problem 55, discern the values of the real integrals

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta \quad \text{and} \quad \int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta - n\theta)\,d\theta.$$

57. Suppose a real function f is continuous on the interval $[a,\, b]$ except at a point c within the interval. Then the **principal value** of the integral is defined by

$$\text{P.V.} \int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \left[\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^b f(x)\,dx \right],\ \varepsilon > 0.$$

Compute the principal value of $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$.

58. Determine whether the integral in Problem 57 converges.

6.6.4 The Argument Principle and Rouché's Theorem

In Problems 59 and 60, use the argument principle in (28) of Theorem 6.20 to evaluate the integral $\displaystyle\oint_C \frac{f'(z)}{f(z)}\,dz$ for the given function f and closed contour C.

59. $f(z) = z^6 - 2iz^4 + (5-i)z^2 + 10$, C encloses all the zeros of f

60. $f(z) = \dfrac{(z - 3iz - 2)^2}{z(z^2 - 2z + 2)^5}$, C is $|z| = \frac{3}{2}$

In Problems 61–64, use the argument principle in (28) of Theorem 6.20 to evaluate the given integral on the indicated closed contour C. You will have to identify $f(z)$ and $f'(z)$.

61. $\displaystyle\oint_C \frac{2z+1}{z^2+z}\,dz$, C is $|z| = 2$

62. $\displaystyle\oint_C \frac{z}{z^2+4}\,dz$, C is $|z| = 3$

63. $\displaystyle\oint_C \cot z\,dz$, C is the rectangular contour with vertices $10+i$, $-4+i$, $-4-i$, and $10-i$.

64. $\displaystyle\oint_C \tan \pi z\,dz$, C is $|z-1| = 2$

65. Use Rouché's theorem (Theorem 6.21) to show that all seven of the zeros of $g(z) = z^7 + 10z^3 + 14$ lie within the annular region $1 < |z| < 2$.

66. **(a)** Use Rouché's theorem (Theorem 6.21) to show that all four of the zeros of $g(z) = 4z^4 + 2(1-i)z + 1$ lie within the disk $|z| < 1$.

(b) Show that three of the zeros of the function g in part (a) lie within the annular region $\frac{1}{2} < |z| < 1$.

67. In the proof of Theorem 6.21, explain how the hypothesis that the strict inequality $|f(z) - g(z)| < |f(z)|$ holds for all z on C implies that f and g cannot have zeros on C.

6.6.5 Summing Infinite Series

68. **(a)** Use the procedure illustrated in Example 8 to obtain the general result

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + a^2} = \frac{1}{2a^2} + \frac{\pi}{2a} \coth a\pi.$$

(b) Use part (a) to verify (47) when $a = 2$.

(c) Find the sum of the series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^2+1}$.

In Problems 69 and 70, use (41) find the sum of the given series.

69. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$

70. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{16k^2+16k+3}$

In Problems 71 and 72, use (43) find the sum of the given series.

71. $\displaystyle\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(4k+1)^2}$

72. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$

73. (**a**) Use (41) to obtain the general result

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k-a)^2} = \frac{\pi^2}{\sin^2 \pi a}$$

where $a \neq 0, \pm 1, \pm 2, \ldots$.

(**b**) Use part (a) to verify your answer to Problem 69.

74. (**a**) Use (43) to obtain the general result

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+a)^2} = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a},$$

where $a \neq 0, \pm 1, \pm 2, \ldots$.

(**b**) Use part (a) to verify your answer to Problem 71

6.7 Applications

In other courses in mathematics or engineering you may have used the **Laplace transform** of a real function f defined for $t \geq 0$,

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt. \tag{1}$$

In the application of (1) we face two problems:

(*i*) *The direct problem*: Given a function $f(t)$ satisfying certain conditions, find its Laplace transform.

When the integral in (1) converges, the result is a function of s. It is common practice to emphasize the relationship between a function and its transform by using a lowercase letter to denote the function and the corresponding uppercase letter to denote its Laplace transform, for example $\mathscr{L}\{f(t)\} = F(s)$, $\mathscr{L}\{y(t)\} = Y(s)$, and so on.

(*ii*) *The inverse problem*: Find the function $f(t)$ that has a given transform $F(s)$.

The function $F(s)$ is called the **inverse Laplace transform** and is denoted by $\mathscr{L}^{-1}\{F(s)\}$.

The Laplace transform is an invaluable aid in solving solve certain kinds of applied problems involving differential equations. In these problems we deal with the transform $Y(s)$ of an unknown function $y(t)$. The determination of $y(t)$ requires the computation of $\mathscr{L}^{-1}\{Y(s)\}$. In the case when $Y(s)$ is a rational function of s, you may recall employing partial fractions, operational properties, or tables to compute this inverse.