

are the real numbers $z = (2n + 1)\pi/2$, $n = 0, \pm 1, \pm 2, \dots$. Since only $-\pi/2$ and $\pi/2$ are within the circle $|z| = 2$, we have

$$\oint_C \tan z \, dz = 2\pi i \left[\operatorname{Res} \left(f(z), -\frac{\pi}{2} \right) + \operatorname{Res} \left(f(z), \frac{\pi}{2} \right) \right].$$

With the identifications $g(z) = \sin z$, $h(z) = \cos z$, and $h'(z) = -\sin z$, we see from (4) that

$$\operatorname{Res} \left(f(z), -\frac{\pi}{2} \right) = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1$$

and
$$\operatorname{Res} \left(f(z), \frac{\pi}{2} \right) = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1.$$

Therefore,
$$\oint_C \tan z \, dz = 2\pi i[-1 - 1] = -4\pi i.$$

Theorem 6.16 is applicable at an essential singularity.



Although there is no nice formula analogous to (1), (2), or (4), for computing the residue at an essential singularity z_0 , the next example shows that Theorem 6.16 is still applicable at z_0 .

EXAMPLE 8 Evaluation by the Residue Theorem

Evaluate $\oint_C e^{3/z} dz$, where the contour C is the circle $|z| = 1$.

Solution As we have seen, $z = 0$ is an essential singularity of the integrand $f(z) = e^{3/z}$ and so neither formulas (1) and (2) are applicable to find the residue of f at that point. Nevertheless, we saw in Example 1 that the Laurent series of f at $z = 0$ gives $\operatorname{Res}(f(z), 0) = 3$. Hence from (5) we have

$$\oint_C e^{3/z} dz = 2\pi i \operatorname{Res}(f(z), 0) = 2\pi i (3) = 6\pi i.$$

EXERCISES 6.5

Answers to selected odd-numbered problems begin on page ANS-20.

In Problems 1–6, use an appropriate Laurent series to find the indicated residue.

1. $f(z) = \frac{2}{(z-1)(z+4)}$; $\operatorname{Res}(f(z), 1)$
2. $f(z) = \frac{1}{z^3(1-z)^3}$; $\operatorname{Res}(f(z), 0)$
3. $f(z) = \frac{4z-6}{z(2-z)}$; $\operatorname{Res}(f(z), 0)$

29. $\oint_C \cot \pi z \, dz$, C is the rectangle defined by $x = \frac{1}{2}$, $x = \pi$, $y = -1$, $y = 1$
30. $\oint_C \frac{2z-1}{z^2(z^3+1)} \, dz$, C is the rectangle defined by $x = -2$, $x = 1$, $y = -\frac{1}{2}$, $y = 1$
31. $\oint_C \left(z^2 e^{1/\pi z} + \frac{ze^z}{z^4 - \pi^4} \right) dz$, $C: 4x^2 + y^2 = 16$
32. $\oint_C \frac{\cos z}{(z-1)^2(z^2+9)} \, dz$, $C: |z-1| = 1$
33. $\oint_C \frac{1}{z^6+1} \, dz$, C is the semicircle defined by $y = 0$, $y = \sqrt{4-x^2}$
34. $\oint_C e^{4/(z-2)} \, dz$, $C: |z-1| = 3$

Focus on Concepts

35. (a) Use series to show that $z = 0$ is a zero of order 2 of $1 - \cos z$.
- (b) In view of part (a), $z = 0$ is a pole of order two of the function $f(z) = e^z/(1 - \cos z)$ and hence has a Laurent series

$$f(z) = \frac{e^z}{1 - \cos z} = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1z + a_2z^2 + \cdots$$

valid for $0 < |z| < 2\pi$. Use series for e^z and $1 - \cos z$ and equate coefficients in the product

$$e^z = (1 - \cos z) \left(\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + \cdots \right)$$

to determine a_{-2} , a_{-1} , and a_0 .

- (c) Evaluate $\oint_C \frac{e^z}{1 - \cos z} \, dz$, where C is $|z| = 1$.
36. Discuss how to evaluate $\oint_C e^{1/z} \sin\left(\frac{1}{z}\right) dz$, where C is $|z| = 1$. Carry out your ideas.
37. Consider the function $f(z) = z^4/(1 - z^{1/2})$, where $z^{1/2}$ denotes the principal branch of the square root function. Discuss and justify your answer: Does f have a pole at $z = 1$? If so, find $\text{Res}(f(z), 1)$.
38. Residues can be used to find coefficients in partial fraction decompositions of rational functions. Suppose that $p(z)$ is a polynomial of degree ≤ 2 and that z_1 , z_2 , and z_3 are distinct complex numbers that are not zeros of $p(z)$. Then

$$f(z) = \frac{p(z)}{(z-z_1)(z-z_2)(z-z_3)} = \frac{A}{z-z_1} + \frac{B}{z-z_2} + \frac{C}{z-z_3}.$$

- (a) Use (1) to show

$$A = \text{Res}(f(z), z_1) = \frac{p(z_1)}{(z_1 - z_2)(z_1 - z_3)}$$

$$B = \text{Res}(f(z), z_2) = \frac{p(z_2)}{(z_2 - z_1)(z_2 - z_3)}$$

$$C = \text{Res}(f(z), z_3) = \frac{p(z_3)}{(z_3 - z_1)(z_3 - z_2)}.$$